Green's Theorem

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Curl and Divergence

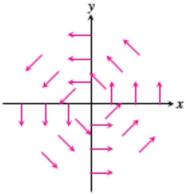
A two-dimensional vector field is a vector-valued function

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$

that one can visualize with a field of arrows.

For example, the below graph is a visualization of the vector field

$$F(x, y) = (y, x).$$



A 2D circulating vector field

Curl and Divergence

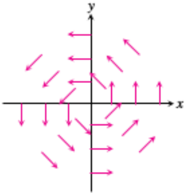
One can think of such a vector field as representing fluid flow in two dimensions, so that

$$\mathrm{F}(x, y)$$

gives the velocity of a fluid at the point (x, y).

In this case, we may call $\mathrm{F}(x, y)$ the velocity field of the fluid.

With this interpretation, the above example illustrates the clockwise circulation of fluid around the origin.



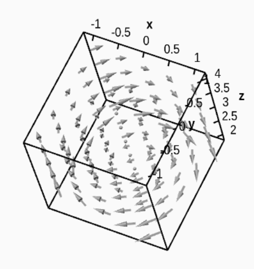
Curl and Divergence

The same interpretation is possible for a three-dimensional fluid flow with velocity represented by a vector field

$$F : \mathbb{R}^3 \to \mathbb{R}^3.$$

In this case, $F(x, y, z)$ is the velocity of the fluid at the point (x, y, z), and we can visualize it as the vector $F(x, y, z)$ positioned a the point (x, y, z).

For example, $F(x, y, z) = (y/z, -x/z, 0)$ can be viewed as fluid circulating around the z-axis.



Curl and Divergence

The connection between the fluid's boundary behavior and its internal behavior is made possible by the notions of divergence and curl.

The divergence of a fluid's velocity field measures the rate at which fluid is being piped into or out of the region at any given point.

The curl measures the fluid's rate of rotation at each point.

Green's theorem in the plane can be used to describe the relationship between the way an incompressible fluid flows along or across the boundary of a plane region and the way it moves inside the region.

Two forms of Green's theorem

Flux-Divergence or Normal Form

Green's theorem states that, under conditions usually met in practice, the outward flux of a vector field across the boundary of a plane region equals the double integral of the divergence of the field over the interior of the region.

Circulation-Curl or Tangential Form

Green's theorem states that, under conditions usually met in practice, the counterclockwise circulation of a field around the boundary of a region equals the double integral of the curl of the field over the region.

Green's Theorem in the Plane

- Green's theorem is one of the great theorems in calculus.
- It is deep and surprising and has far-reaching consequences.
- In pure mathematics, it ranks in importance with the Fundamental Theorem of Calculus.
- In applied mathematics, the generalizations of Green's theorem to three dimensions provide the foundation for theorems about electricity, magnetism, and fluid flow.

Green's theorem applies to any vector field satisfying certain mathematical conditions. It does not depend for its validity on the field's having a particular physical interpretation.

However, we discuss the theorem in terms of velocity fields of fluid flows because fluid flows are easy to picture.

Green's theorem (general form) states that a line integral around the boundary of a plane region R can be computed as a double integral over R. More precisely, if R is a "nice" region in the plane and C is the boundary of R with C oriented so that R is always on the left-hand side as one goes around C (this is the positive orientation of C), then

$$\oint_C M\ dx + N\ dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy$$

if the partial derivatives of M and N are continuous on R.

Flux Density at a Point : Divergence

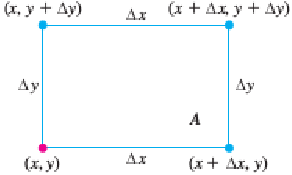
The flux density of a vector field at a point, which in mathematics is called the **divergence** of the vector field. We obtain it in the following way.

Suppose that

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

is the velocity field of a fluid flow in the plane and that the **first partial derivations of M and N are continuous at each point of a region R.**

Let (x, y) be a point in R and let A be a small rectangle with one corner at (x, y) that, along with its interior, lies entirely in R.



Flux Density at a Point : Divergence

The sides of the rectangle, parallel to the coordinate axes, have lengths of Δx and Δy.

The rate at which fluid leaves the rectangle across the bottom edge is approximately

$$\mathrm{F}(x, y).(-\mathrm{j})\Delta x = -N(x, y)\ \Delta x.$$

This is the scalar component of the velocity at (x, y) in the direction of the outward normal times the length of the segment.

For example, if the velocity is in meters per second, the exit rate will be in meters per second times meters or square meters per second (m^2/s).

Flux Density at a Point : Divergence

The rates at which the fluid crosses the other three sides in the directions of their outward normals can be estimated in a similar way.

We have

Top : $\mathrm{F}(x, y+\Delta y).(-\mathrm{j})\Delta x = N(x, y+\Delta y)\ \Delta x$

Bottom : $\mathrm{F}(x, y).(-\mathrm{j})\Delta x = -N(x, y)\ \Delta x$

Right : $\mathrm{F}(x+\Delta x, y).(\mathrm{i})\Delta y = M(x+\Delta x, y)\ \Delta y$

Left : $\mathrm{F}(x, y).(-\mathrm{i})\Delta y = -M(x, y)\ \Delta y.$

Combining opposite pairs gives

Top and bottom : $\big(N(x, y+\Delta y) - N(x, y)\big)\ \Delta x \approx \big(\frac{\partial N}{\partial y}\Delta y\big)\Delta x$

Right and left : $\big(M(x+\Delta x, y) - M(x, y)\big)\ \Delta y \approx \big(\frac{\partial M}{\partial x}\Delta x\big)\Delta y.$

Flux Density at a Point : Divergence

Adding the above relations gives :

Flux across rectangle boundary $\approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \, \Delta y$.

We now divide by $\Delta x \, \Delta y$ to estimate **the total flux per unit area or flux density** for the rectangle :

$$\frac{\text{Flux across rectangle boundary}}{\text{Rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right).$$

Finally, we let Δx and Δy approach zero to define what we call the **flux density** of F at the point (x, y).

We call the flux density, the divergence of F and it is denoted by div F.

Flux Density at a Point : Divergence

The flux density or divergence of a vector field

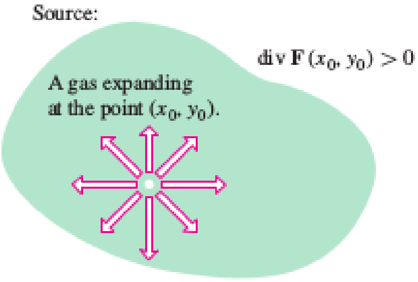
$$\mathsf{F} = M\mathsf{i} + N\mathsf{j}$$

at the point (x, y) is

$$\text{div } \mathsf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

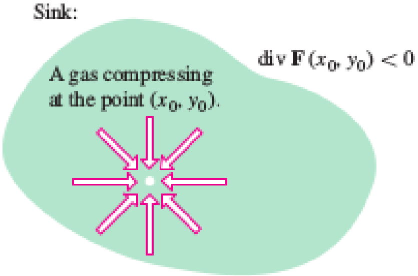
Flux Density at a Point : Divergence

Intuitively, if water is flowing into a region through a small hole at the point (x_0, y_0), the lines of flow would diverge there (hence the name) and, since water would be flowing out of a small rectangle about (x_0, y_0), the divergence of F at (x_0, y_0) would be positive.



Flux Density at a Point : Divergence

If the water is draining out instead of flowing in, the divergence would be negative.

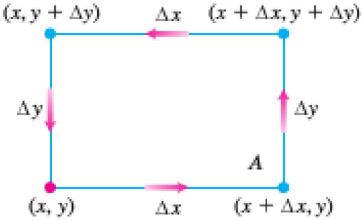


Circulation Density at a Point : Curl

The circulation density of a vector field at a point, which in mathematics is called the **curl** of the vector field. We obtain it in the following way.

Suppose that $\mathrm{F}(x, y) = M(x, y)\mathrm{i} + N(x, y)\mathrm{j}$ is the velocity field of a fluid flow in the plane and that **the first partial derivations of** M **and** N **are continuous at each point of a region** R**.**

Let (x, y) be a point in R and let A be a small rectangle with one corner at (x, y) that, along with its interior, lies entirely in R.



Circulation Density at a Point : Curl

The sides of the rectangle, parallel to the coordinate axes, have lengths of Δx and Δy.

The counterclockwise circulation of F around the boundary of A is the sum of flow rates along the sides.

The rate of flow (flow rate) along the bottom edge is approximately

$$\mathrm{F}(x, y).\mathrm{i}\Delta x = M(x, y)\ \Delta x.$$

This is the scalar component of the velocity $\mathrm{F}(x, y)$ in the direction of the tangent vector i times the length of the segment.

The **rates of flow** along the other sides in the counterclockwise direction are expressed in a similar way.

Circulation Density at a Point : Curl

In all, we have

Top : $\mathrm{F}(x, y+\Delta y).(-\mathrm{i})\Delta x = -M(x, y+\Delta y)\ \Delta x$

Bottom : $\mathrm{F}(x, y).(\mathrm{i})\Delta x = M(x, y)\ \Delta x$

Right : $\mathrm{F}(x+\Delta x, y).(\mathrm{j})\Delta y = N(x+\Delta x, y)\ \Delta y$

Left : $\mathrm{F}(x, y).(-\mathrm{j})\Delta y = -N(x, y)\ \Delta y.$

Combining opposite pairs gives

Top and bottom : $-\big(M(x, y+\Delta y) - M(x, y)\big)\ \Delta x \approx -\big(\frac{\partial M}{\partial y}\Delta y\big)\Delta x$

Right and left : $\big(N(x+\Delta x, y) - N(x, y)\big)\ \Delta y \approx \big(\frac{\partial N}{\partial x}\Delta x\big)\Delta y.$

Circulation Density at a Point : Curl

Adding the above relations and dividing by $\Delta x \Delta y$ gives an estimate of the circulation density for the rectangle :

$$\frac{\text{Circulation around rectangle}}{\text{Rectangle area}} \approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

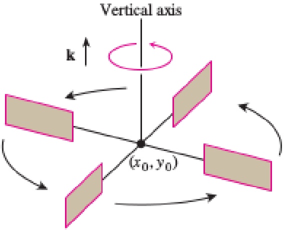
Finally, we let Δx and Δy approach zero to define what we call the **circulation density** of F at the point (x, y).

We call the circulation density, the curl of F and it is denoted by curl F.

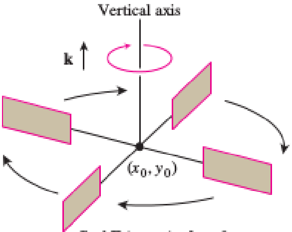
Circulation Density at a Point : Curl

The circulation density or curl of a vector field $\mathsf{F} = M\mathsf{i} + N\mathsf{j}$ at the point (x, y) is curl $\mathsf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

If water is moving about a region in the xy-plane in a thin layer, then the cirulation, or curl, at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel will spin if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane.



Curl **F** $(x_0, y_0) \cdot \mathbf{k} > 0$
Counterclockwise circulation



Curl **F** $(x_0, y_0) \cdot \mathbf{k} < 0$
Clockwise circulation

Two forms of Green's theorem

Flux-Divergence or Normal Form

Green's theorem states that, under suitable conditions, the outward flux of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve.

Green's Theorem (Flux-Divergence or Normal Form)

The outward flux of a field $\mathrm{F} = M\mathrm{i} + N\mathrm{j}$ across a simple closed curve C equals the double integral of divF over the region R enclosed by C.

$$\begin{aligned} \oint \mathrm{F.n}\ ds &= \oint M\ dy - N\ dx \\ &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\ dy. \end{aligned}$$

Two forms of Green's theorem

Circulation-Curl or Tangential Form

Green's theorem states that, under conditions usually met in practice, the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the curl of the field over the region enclosed by the curve.

Green's Theorem (Circulation-Curl or Tangential Form)

The counterwise circulation of a field $\mathrm{F} = M\mathrm{i} + N\mathrm{j}$ around a simple closed curve C in the plane equals the double integral of curlF over the region R enclosed by C.

$$\begin{aligned} \oint \mathrm{F.T}\ ds &= \oint M\ dx + N\ dy \\ &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\ dy. \end{aligned}$$

Assumptions for Green's theorem to hold

When we wish to apply Green's theorem, we need two kinds of assumptions for Green's theorem to hold.

1. We need conditions on M and N to ensure the existence of the integrals.
   The usual assumptions are that M, N, and their first partial derivatives are continuous at every point of some open region containing C and R.
2. We need geometric conditions on the curve C. It must be simple, closed, and made up of pieces along which we can integrate M and N.
   The usual assumptions are that C is piecewise smooth.

Calculating Area with Green's Theorem

Green's Theorem Area Formula : If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

$$\text{Area of } R = \frac{1}{2} \oint_C x\,dy - y\,dx$$

George Green

George Green (1793-1841) was a self-taught scientist in Nottingham, England.

Green's work on the mathematical foundations of gravitation, electricity, and magnetism was published privately in 1828 in a short book entitled *"An Essay on the Application of Mathematical Analysis to Electricity and Magnetism."*

George Green

The book sold all of fifty-two copies (fewer than one hundred were printed), the copies going mostly to Greeen's patrons and personal friends.

A few weeks before Green's death in 1841, William Thomson noticed a reference to Green's book and in 1845 was finally able to locate a copy.

Excited by what he read, Thomson shared Green's ideas with other scientists and had the book republished in a series of journal articles.

Green's mathematics provided the foundation on which Thomson, Stokes, Rayleigh, and Maxwell built the present-day theory of electrogagnetism.

Exercise 1.

In the following exercises, verify the conclusion of Green's Theorem for the field $\mathrm{F} = M\mathrm{i} + N\mathrm{j}$*. Take the domains of integration in each case to be the disk* $R : x^2 + y^2 \leq a^2$ *and its bounding circle* $C : \mathrm{r} = (a\cos t)\mathrm{i} + (a\sin t)\mathrm{j}$, $0 \leq t \leq 2\pi$.

1. $\mathrm{F} = -y\mathrm{i} + x\mathrm{j}$
2. $\mathrm{F} = -x^2y\mathrm{i} + xy^2\mathrm{j}$

1. $M = -y = -a\sin\ t, N = x = a\ \cos\ t, dx = -a\ \sin\ t\ dt\ , dy = a\ \cos\ t\ dt\ \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = -1, \frac{\partial N}{\partial x} = 1,$ and $\frac{\partial N}{\partial y} = 0;$
   Equations(3); $\oint_c M\ dy - N\ dx = \int_0^{2\pi}[(-a\ sin\ t)(a\ \cos\ t) - (a\ \cos\ t)(-a\ \sin\ t)]dt = \int_0^{2\pi} 0\ dt = 0; \int_R \int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\ dy = \int_R \int 0\ dx\ dy = 0,$ Flux
   Equation (4):$\oint_c M\ dx + N\ dy = \int_0^{2\pi}[(-a\ \sin\ t)(-a\ \sin\ t) - (a\ \cos\ t)(a\ \cos\ t)]dt = \int_0^{2\pi} a^2 dt = 2\pi a^2; \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy = \int_{-a}^{a} \int_{-c}^{\sqrt{a^2-x^2}} 2\ dy\ dx = \int_{-a}^{a} 4\sqrt{a^2 - x^2} dx = 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}\right]_{-a}^{a} = 2a^2 \left(\frac{\pi}{2} + \frac{\pi}{2} = 2a^2\pi, \right)$Circulation
2. $M = -x^2y = -a^3\cos^2 t, N = xy^2 = a^3\cos\ t\ sin^2 t, dx = -a\ \sin\ t\ dt, dy = a\ \cos\ t\ dt\ \Rightarrow \frac{\partial M}{\partial x} = -2xy, \frac{\partial M}{\partial y} = -x^2, \frac{\partial N}{\partial x} = y^2,$ and $\frac{\partial N}{\partial y} = 2xy;$
   Equation (3):
   $\oint M\ dy - N\ dx = \int_0^{2\pi}(-a^4\cos^3 t\sin\ t + a^4\cos\ t\ \sin^3 t) = \left[\frac{a^4}{4}\cos^4 t + \frac{a^4}{4}\sin^4 t\right]_0^{2\pi} = 0;$
   $\int_R \int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\ dy = \int_R \int (-2xy + 2xy) dx\ dy = 0,$Flux
   Equation
   (4):$\oint_c M\ dx + N\ dy = \int_0^{2\pi}(a^4\cos^2 t\ \sin^2 t + a^4\cos^2 t\ \sin^2 t)dt = \int_0^{2\pi}(2a^4\cos^2 t\ \sin^2 t)dt = \int_0^{2\pi} \frac{1}{2}a^4\sin^2 2t\ dt = \frac{a^4}{4}\int_0^{4\pi}\sin^2 u\ du = \frac{a^4}{4}\left[\frac{u}{2} - \frac{\sin\ 2u}{4}\right]_0^{4\pi} = \frac{\pi a^4}{2}; \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy = \int_R \int (y^2 + x^2) dx\ dy = \int_0^{2\pi}\int_0^a r^2 \cdot r\ dr\ d\theta = \int_0^{2\pi} \frac{a^4}{4} d\theta = \frac{\pi a^4}{2},$ Circulation

Exercise 2.

In the following exercises, use Green's Theorem to find the counterclockwise circulation and outward flux for the field F *and curve* C.

1. $\mathrm{F} = (x^2 + 4y)\mathrm{i} + (x + y^2)\mathrm{j}$
   C : *The square bounded by* $x = 0$, $x = 1$, $y = 0$, $y = 1$
2. $\mathrm{F} = (x^2 - y^2)\mathrm{i} + 2xy\mathrm{j}$
   C : *Boundary of the region formed by the curves* $y = x^2$ *and* $x = y^2$
3. $\mathrm{F} = (x + e^x \sin y)\mathrm{i} + (x + e^x \cos y)\mathrm{j}$
   C :*The right-hand loop of the lemniscate* $r^2 = \cos 2\theta$
4. $\mathrm{F} = \left(\tan^{-1} \frac{y}{x}\right) \mathrm{i} + \ln(x^2 + y^2)\mathrm{j}$
   C : *The boundary of the region defined by the polar coordinate inequalities* $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$

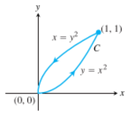
Solution for Exercise 2

1. $M = x^2+4y$, $N = x+y^2 \Rightarrow \frac{\partial M}{\partial x} = 2x$, $\frac{\partial M}{\partial y} = 4$, $\frac{\partial N}{\partial x} = 1$, $\frac{\partial N}{\partial y} = 2y \Rightarrow$ Flux $= \int_R \int (2x+2y)dx\, dy = \int_0^1 \int_0^1 (2x+2y)dx\, dy = \int_0^1 [x^2+2xy]_0^1 dy = \int_0^1 (1+2y)dy = [y+y^2]_0^1 = 2$; Circ $= \int_R \int (1-4)dx\, dy = \int_0^1 \int_0^1 -3\, dx\, dy = -3$
2. $M = x+y$, $N = -(x^2+y^2) \Rightarrow \frac{\partial M}{\partial x} = 1$, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -2x$, $\frac{\partial N}{\partial y} = -2y \Rightarrow$ Flux $= \int_R \int (1-2y)dx\, dy = \int_0^1 \int_0^x (1-2y)dy\, dx = \int_0^1 (x-x^2)dx = \frac{1}{6}$; Circ $= \int_R \int (-2x-1)dx\, dy = \int_0^1 \int_0^x (-2x-1)dy\, dx = \int_0^1 (-2x^2-x)dx = -\frac{7}{6}$
3. $M = x+e^x \sin\ y$, $N = x+e^x \cos y \Rightarrow \frac{\partial M}{\partial x} = 1+e^x \sin\ y$, $\frac{\partial M}{\partial y} = e^x \cos\ y$, $\frac{\partial N}{\partial x} = 1+e^x \cos y$, $\frac{\partial N}{\partial y} = -e^x \sin\ y \Rightarrow$ Flux $= \int_R \int dx\, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{cos 2\theta}} r\, dr\, d\theta = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} \cos\ 2\theta\right) d\theta = \left[\frac{1}{4} \sin\ 2\theta\right]_{-\pi/4}^{\pi/4} = \frac{1}{2}$;
Circ $= \int_R \int (1+e^x \cos\ y - e^x \cos y)dx\, dy = \int_R \int dx\, dy = \int_{-\pi/4}^{\pi/4} \int_0^{\sqrt{\cos 2\theta}} r\, dr\, d\theta$
$= \int_{-1\pi/4}^{\pi/4} (\frac{1}{2} \cos\ 2\theta)d\theta = \frac{1}{2}$
4. $M = \tan^{-1} \frac{y}{x}$, $N = \ln(x^2+y^2) \Rightarrow \frac{\partial M}{\partial x} = \frac{-y}{x^2+y^2}$, $\frac{\partial M}{\partial y} = \frac{x}{x^2+y^2}$, $\frac{\partial N}{\partial x} = \frac{2x}{x^2+y^2}$, $\frac{\partial N}{\partial y} = \frac{2y}{x^2+y^2}$
$\Rightarrow$ Flux $= \int_R \int \left(\frac{-y}{x^2+y^2} + \frac{2y}{x^2+y^2}\right) dx\, dy = \int_0^\pi \int_1^2 \left(\frac{r \sin \theta}{r^2}\right) r\, dr\, d\theta$
$= \int_0^\pi \sin\ \theta\, d\theta = 2$; Circ $= \int_R \int \left(\frac{2x}{x^2+y^2} - \frac{x}{x^2+y^2}\right) dx\, dy = \int_0^\pi \int_1^2 \left(\frac{r \cos \theta}{r^2}\right) r\, dr\, d\theta = \int_0^\pi \cos\theta\, d\theta = 0$

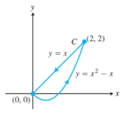
Exercise 3.

In the following exercises, use Green's Theorem to find the counterclockwise circulation and outward flux for the field F *and curve* C.

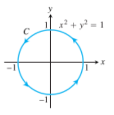
1. $\mathsf{F} = (xy + y^2)\mathsf{i} + (x - y)\mathsf{j}$



2. $\mathsf{F} = x^3y^2\mathsf{i} + \frac{1}{2}x^4y\mathsf{j}$



3. $\mathsf{F} = \frac{x}{1+y^2}\mathsf{i} + (\tan^{-1} y)\mathsf{j}$



Solution for Exercise 3

1. $M = xy + y^2, N = x - y \Rightarrow \frac{\partial M}{\partial x} = y, \frac{\partial M}{\partial y} = x + 2y, \frac{\partial N}{\partial x} = 1, \frac{\partial N}{\partial y} = -1 \Rightarrow \text{Flux} = \int_R \int (y + (-1))dy\ dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (y-1)dy\ dx = \int_0^1 (\frac{1}{2}x - \sqrt{x} - \frac{1}{2}x^4 + x^2)dx = -\frac{11}{60}$; $\text{Circ} = \int_R \int (1 - (x + 2y))dy\ dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (1 - x - 2y)dy\ dx = \int_0^1 (\sqrt{x} - x^{3/2} - x - x^2 + x^3 + x^4)dx = -\frac{7}{60}$
2. $M = x^3y^2, N = \frac{1}{2}x^4y \Rightarrow \frac{\partial M}{\partial x} = 3x^2y^2, \frac{\partial M}{\partial y} = 2x^3y = \frac{\partial N}{\partial x} = 2x^3y, \frac{\partial N}{\partial y} = \frac{1}{2}x^4 \Rightarrow \text{Flux} = \int_R \int (3x^2y^2 + \frac{1}{2}x^4)dy\ dx = \int_0^2 \int_{x^2-x}^{x} (3x^2y^2 + \frac{1}{2}x^4)dy\ dx = \int_0^2 (3x^5 - \frac{7}{2}x^6 + 3x^7 - x^8)dx = \frac{64}{9}$; $\text{Flux} = \int_R \int (2x^3y - 2x^3y)dy\ dx = 0$
3. $M = \frac{x}{1+y^2}, N = \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = \frac{1}{1+y^2}, \frac{\partial M}{\partial y} = \frac{-2xy}{(1+y^2)^2}, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = \frac{1}{1+y^2} \Rightarrow \text{Flux} = \int_R \int \left(\frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dx\ dy = \int_1^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{1+y^2} dx\ dy = \int_1^1 \frac{4\sqrt{1-y^2}}{1+y^2} dx = 4\pi\sqrt{2} - 4\pi$; $\text{Circ} = \int_R \int \left(0 - \left(\frac{-2xy}{(1+y^2)^2}\right)\right) dy\ dx = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{2xy}{(1+y^2)^2}\right) dy\ dx = \int_{-1}^1 (0)dx = 0$

Exercise 4.

1. *Find the counterclockwise circulation and outward flux of the field* $\mathrm{F} = xy\mathrm{i} + y^2\mathrm{j}$ *around and over the boundary of the region enclosed by the curves* $y = x^2$ *and* $y = x$ *in the first quadrant.*
2. *Find the counterclockwise circulation and the outward flux of the field* $\mathrm{F} = (-\sin y)\mathrm{i} + (x \cos y)\mathrm{j}$ *around and over the square cut from the first quadrant by the lines* $x = \pi/2$ *and* $y = \pi/2$.

Solution for Exercise 4

1. $M = xy, N = y^2 \Rightarrow \frac{\partial M}{\partial x} = y, \frac{\partial M}{\partial y} = x, \frac{\partial N}{\partial x} = 0, \frac{\partial N}{\partial y} = 2y \Rightarrow \text{Flux} = \int_R \int (y + 2y) dy\ dx = \int_0^1 \int_{x^2}^x 3y\ dy\ dx = \int_0^1 \left(\frac{3x^2}{2} - \frac{3x^4}{2} \right) dx = \frac{1}{5}$; $\text{Circ} = \int_R \int -x\ dy\ dx = \int_0^1 \int_{x^2}^x -x\ dy\ dx = \int_0^1 (-x^2 + x^3) dx = -\frac{1}{12}$
2. $M = -\sin\ y, N = x\ \cos\ y \Rightarrow \frac{\partial M}{\partial x} = 0, \frac{\partial M}{\partial y} = -\cos\ y, \frac{\partial N}{\partial x} = \cos\ y, \frac{\partial N}{\partial y} = -x\ \sin\ y \Rightarrow$ $\text{Flux} = \int_R \int (-x\ \sin\ y) dx\ dy = \int_0^{\pi/2} \int_0^{\pi/2} (-x\ \sin\ y) dx\ dy = \int_0^{\pi/2} \left(-\frac{\pi^2}{8} \sin\ y \right) dy = -\frac{\pi^2}{8}$; $\text{Circ} = \int_R \int [\cos\ y - (-\cos\ y)] dx\ dy = \int_0^{\pi/2} \int_0^{\pi/2} 2\ \cos\ y\ dx\ dy = \int_0^{\pi/2} \pi\ \cos\ y\ dy = [\pi\ \sin\ y]_0^{\pi/2} = \pi$

Exercises

Exercise 5.

1. *Find the outward flux of the field*

$$F = \left(3xy - \frac{x}{1+y^2}\right)\mathrm{i} + (e^x + \tan^{-1} y)\mathrm{j}$$

   across the cardioid $r = a(1+\cos\theta)$, $a > 0$.

2. *Find the counterclockwise circulation of* $F = (y + e^x \ln y)\mathrm{i} + (e^x/y)\mathrm{j}$ *around the boundary of the region that is bounded above by the curve* $y = 3 - x^2$ *and below by the curve* $y = x^4 + 1$.

Solution for Exercise 5

1. $M = 3xy - \frac{x}{1+y^2}, N = e^x + \tan^{-1} y \Rightarrow \frac{\partial M}{\partial x} = 3y - \frac{1}{1+y^2}, \frac{\partial N}{\partial y} = \frac{1}{1+y^2} \Rightarrow \text{Flux} =$
$\int_R \int \left(3y - \frac{1}{1+y^2} + \frac{1}{1+y^2}\right) dx\ dy = \int_R \int 3y\ dx\ dy = \int_0^{2\pi} \int_0^{a(1+\cos\theta)} (3r\ \sin\ \theta) r\ dr\ d\theta =$
$\int_0^{2\pi} a^3(1+\cos\theta)^3(\sin\theta) d\theta = \left[-\frac{a^3}{4}(1+\cos\theta)^4\right]_0^{2\pi} = -4a^3 - (-4a^3) = 0$
2. $M = y + e^x \ln\ y, N = \frac{e^x}{y} \Rightarrow \frac{\partial M}{\partial y} = 1 + \frac{e^x}{y}, \frac{\partial N}{\partial x} = \frac{e^x}{y} \Rightarrow \text{Circ} =$
$\int_R \int \left[\frac{e^x}{y} - \left(1 + \frac{e^x}{y}\right)\right] dx\ dy = \int_R \int (-1) dx\ dy = \int_{-1}^{1} \int_{x^4+1}^{3-x^2} -dy\ dx =$
$-\int_{-1}^{1} [(3-x^2) - (x^4+1)] dx = \int_{-1}^{1} (x^4 + x^2 - 2) dx = -\frac{44}{15}$

Exercise 6.

Find the work done by $\mathrm{F} = (4x - 2y)\mathrm{i} + (2x - 4y)\mathrm{j}$ *in moving a particle once counterclockwise around the curve*
C : *The circle* $(x-2)^2 + (y-2)^2 = 4$.

Solution for Exercise 6

$M = 4x - 2y, N = 2x - 4y \Rightarrow \frac{\partial M}{\partial y} = -2, \frac{\partial N}{\partial x} = 2 \Rightarrow$ work $=$
$\oint_c (4x - 2y)dx + (2x - 4y)dy = \int_R \int [2 - (-2)]dx\ dy = 4 \int_R \int dx\ dy =$
$4(\text{Area of the circle}) = 4(\pi \cdot 4) = 16\pi$

Using Green's Theorem

Exercise 7.

Apply Green's Theorem to evaluate the integrals in the following exercises.

1. $\oint_C (y^2\, dx + x^2\, dy)$
   C : *The triangle bounded by* $x = 0, x + y = 1, y = 0.$
2. $\oint_C (6y + x)\, dx + (y + 2x)\, dy$
   C : *The circle* $(x-2)^2 + (y-3)^2 = 4.$

Solution for Exercise 7

1. $M = y^2, N = x^2 \Rightarrow \frac{\partial M}{\partial y} = 2y, \frac{\partial N}{\partial x} = 2x \Rightarrow \oint_c y^2 dx + x^2 dy = \int_R \int (2x - 2y) dy\ dx = \int_0^1 \int_0^{1\ x} (2x - 2y) dy\ dx = \int_0^1 (-3x^2 + 4x - 1) dx = [-x^3 + 2x^2 - x]_0^1 = -1 + 2 - 1 = 0$
2. $M = 6y + x, N = y + 2x \Rightarrow \frac{\partial M}{\partial y} = 6, \frac{\partial N}{\partial x} = 2 \Rightarrow \oint_c (6y + x) dx + (y + 2x) dy = \int_R \int (2 - 6) dy\ dx = -4(\text{Area of the circle}) = -16\pi$

Exercise 8.

Use the Green's Theorem area formula given above to find the areas of the regions enclosed by the curves.

1. *The ellipse* $\mathrm{r}(t) = (a\cos t)\mathrm{i} + (b\sin t)\mathrm{j}, \quad 0 \leq t \leq 2\pi$.
2. *One arch of the cycloid* $x = t - \sin t,\ y = 1 - \cos t$.
3. *Find the area of the region bounded by* $y^2 = 4x$ *and* $x^2 = 4y$ *by Green's theorem.*

Solution for Exercise 8

1. $M = x = a\ \cos\ t, N = y = b\ \sin\ t \Rightarrow dx = -a\ \sin\ t\ dt, dy = b\ \cos\ t\ dt \Rightarrow$ Area $=$ $\frac{1}{2}\oint_c x\ dy - ydx = \frac{1}{2}\int_0^{2\pi}(ab\ \cos^2 t + ab\ \sin^2 t)dt = \frac{1}{2}\int_0^{2\pi} ab\ dt = \pi ab$
2. $C_1 : M = x = t, N = y = 0 \Rightarrow dx = dt, dy = 0; C_2 : M = x = (2\pi - t) - \sin(2\pi - t) = 2\pi - t + \sin t, N = y = 1 - \cos(2\pi - t) = 1 - \cos\ t \Rightarrow dx = (\cos\ t - 1)dt, dy = \sin\ t\ dt$
   $\Rightarrow$ Area $= \frac{1}{2}\oint_c x\ dy - y\ dx = \frac{1}{2}\oint_{C_1} x\ dy - y\ dx + \frac{1}{2}\oint_{C_2} x\ dy - y\ dx$
   $= \frac{1}{2}\int_0^{2\pi}(0)dt + \frac{1}{2}\int_0^{2\pi}[(2\pi - t + \sin\ t)(\sin\ t) - (1 - \cos\ t)(\cos\ t - 1)]dt =$
   $-\frac{1}{2}\int_0^{2\pi}(2\ \cos\ t + t\ \sin\ t - 2 - 2\pi \sin\ t)dt = -\frac{1}{2}[3\ \sin\ t - t\ cos\ t - 2t - 2\pi \cos\ t]_0^{2\pi} = 3\pi$
3. (a) $M = f(x), N = g(y) \Rightarrow \frac{\partial M}{\partial y} = 0, \Rightarrow \frac{\partial N}{\partial x} = 0 \Rightarrow \oint_c f(x)dx + g(y)dy =$ $\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy = \int_R\int 0\ dx\ dy = 0$
   (b) $M = ky, N = hx \Rightarrow \frac{\partial M}{\partial y} = k, \Rightarrow \frac{\partial N}{\partial x} = h \Rightarrow \oint_c ky\ dx + hx\ dy =$ $\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\ dy = \int_R\int(h-k)dx\ dy =$ $(h - k)$(Area of the region)

Exercise 9.

Show that the value of

$$\oint_C xy^2\,dx + (x^2y + 2x)\,dy$$

around any square depends only on the area of the square and not on its location in the plane.

Solution for Exercise 9

$M = xy^2, N = x^2y + 2x \Rightarrow \frac{\partial M}{\partial y} = 2xy, \frac{\partial N}{\partial x} = 2xy + 2 \Rightarrow$
$\oint_c xy^2 dx + (x^2y + 2x)dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\ dy =$
$\int_R \int (2xy + 2 - 2xy) dx\ dy = 2 \int_R \int dx\ dy = 2$ times the area of the square

Exercise 10.

1. *What is special about the integral*

$$\oint_C 4x^3 y\, dx + x^4\, dy?$$

   Give reasons for your answer.

2. *What is special about the integral*

$$\oint_C -y^3\, dy + x^3\, dx?$$

   Give reasons for your answer.

Solution for Exercise 10

1. The integral is 0 for any simple closed plane curve C. The reasoning: By the tangential form of Green's Theorem, with $M = 4x^3y$ and $N = x^4$, $\oint_c 4x^3y\ dx + x^4 dy =$ $\int_R \int \left[\frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3y)\right] dx\ dy = \int_R \int \underbrace{(4x^3 - 4x^3)}_{0} dx\ dy = 0$

2. The integral is 0 for any simple closed curve C. The reasoning: By the normal form of Green's theorem, with
$M = x^3$ and $N = -y^3$, $\oint_c -y^3 dy + x^3 dx = \int_R \int \left[\underbrace{\frac{\partial}{\partial x}(-y^3)}_{0} - \underbrace{\frac{\partial}{\partial y}(x^3)}_{0}\right] dx\ dy = 0$

Exercise 11.

1. Area as a line integral : *Show that if R is a region in the plane bounded by a piecewise smooth, simple closed curve C, then*

$$\text{Area of } R = \oint_C x\, dy = -\oint_C y\, dx.$$

2. Definite integral as a line integral : *Suppose that a nonnegative function $y = f(x)$ has a continuous first derivative on $[a, b]$. Let C be the boundary of the region in the xy-plane that is bounded below by the x-axis, above by the graph of f, and on the sides by the lines $x = a$ and $x = b$ Show that*

$$\int_a^b f(x)\, dx = -\oint_C ydx.$$

Solution for Exercise 11

1. Let $M = x$ and$N = 0 \Rightarrow \frac{\partial M}{\partial x} = 1$
   and $\frac{\partial N}{\partial y} = 0 \Rightarrow \oint_c M\ dy - N\ dx = \int_R \int \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\ dy \Rightarrow \oint_c x\ dy =$
   $\int_R \int (1+0) dx\ dy \Rightarrow$ Area of$R = \int_R \int dx\ dy = \oint_c x\ dy$;
   similarly, $M = y$ and $N = 0 \Rightarrow \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 0 \Rightarrow \oint_c M\ dx + N\ dy =$
   $\int_R \int \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} \right) dy\ dx \Rightarrow \oint_c y\ dx = \int_R \int (0-1) dy\ dx \Rightarrow -\oint_c y\ dx = \int_R \int dx\ dy =$
   Area of$R$
2. $\int_a^b f(x) dx =$ Area of $R = -\oint_c y\ dx$, from the above exercise.

Exercise 12.

Let A be the area and $\overline{x}$ the x-coordinate of the centroid of a region R that is bounded by a piecewise smooth, simple closed curve C in the xy-plane. Show that

$$\frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\overline{x}.$$

Solution for Exercise 12

Let
$\delta(x,y)=1 \Rightarrow \overline{x}=\frac{M_y}{M}=\frac{\int_R\int x\ \delta(x,y)dA}{\int_R\int \delta(x,y)dA}=\frac{\int_R\int x\ dA}{\int_R\int dA}=\frac{\int_R\int x\ dA}{A} \Rightarrow A\overline{x}=$ $\int_R\int x\ dA=\int_R\int (x+0)dx\ dy=\oint_c \frac{x^2}{2}dy, A\overline{x}=\int_R\int x\ dA=\int_R\int (0+$ $x)dx\ dy=-\oint_c xy\ dx,$ and $A\overline{x}=\int_R\int x\ dA=\int_R\int \left(\frac{2}{3}x+\frac{1}{3}x\right)dx\ dy=$ $\oint_c \frac{1}{3}x^2dy-\frac{1}{3}xy\ dx=\frac{1}{2}\oint_c x^2dy=-\oint_c xy\ dx=\frac{1}{3}\oint_c x^2dy-xy\ dx=A\overline{x}$

Maximizing work

Exercise 13.

1. *Among all smooth, simple closed curves in the plane, oriented counterclockwise, find the one along which the work done by*

$$\mathsf{F} = \left(\frac{1}{4}x^2y + \frac{1}{3}y^3\right)\mathsf{i} + x\mathsf{j}$$

is greatest. (Hint: Where is (curl F*)·*k *positive?)*

2. Curl component of conservative fields : *Can anything be said about the curl component of a conservative two-dimensional vector field? Give reasons for your answer.*

Solution for Exercise 13

1. $M = \frac{1}{4}x^2y + \frac{1}{3}y^3, N = x \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{4}x^2 + y^2, \frac{\partial N}{\partial x} = 1 \Rightarrow \text{Curl} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (\frac{1}{4}x^2 + y^2) > 0$ in the interior of the ellipse $\frac{1}{4}x^2 + y^2 = 1 \Rightarrow$ work $= \int_c \mathbf{F} \cdot d\mathbf{r} = \int_R \int (1 - \frac{1}{4}x^2 - y^2) dx\, dy$ will be maximized on the region $R = \{(x, y) | \text{curl } \mathbf{F}\} \geq 0$ or over the region enclosed by $1 = \frac{1}{4}x^2 + y^2$
2. The curl of a conservative two-dimensional field is zero. The reasoning: A two -dimensional field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ can be considered to be the restriction to the xy-plane of a three-dimensional field whose k component is zero, and whose **i** and **j** components are independent of z. For such a field to be conservative, we must have $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ by the component test in Section 16.3 $\Rightarrow$ curl $\mathbf{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

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